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Infinite-component spin-glass model in the low-temperature phase

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Abstract. The Edwards-Anderson spin-glass model with m-component spins is studied in the low-temperature phase within the context of a Q^3 theory and the limit $m \to \infty$ is taken. Perturbation theory for the order parameter reveals at one-loop order the following unusual properties: (i) the perturbation expansion fails due to the appearance of infrared singularities for dimensionalities $d \leqslant 8$ where $d_c = 8$ is also the upper critical dimension of the theory, and (ii) for $d = 8$ the expected terms involving $\ln(T_c - T)$ are absent thus implying a breakdown of scaling.

1. Introduction

In the theory of phase transitions, the study of certain simplifying limits has long been fashionable. For the case of interacting spin systems, for which a generic model is the classical Heisenberg model with Hamiltonian

$$\mathscr{H} = -\tfrac{1}{2}\sum_{ij} J_{ij}\, \boldsymbol{S}_i \cdot \boldsymbol{S}_j$$

there is a much studied limit in which the dimensionality m of the spin space tends to infinity independently of the spatial dimensionality d. This limit often leads to considerable simplifications, and expansions about it, i.e. in powers of $1/m$, have been used with considerable success in the theory of critical phenomena (Ma 1976 and references therein).

In this paper we consider the $m \to \infty$ limit of the Edwards-Anderson spin-glass model in which the spins $\boldsymbol{S}_i$ are m-dimensional vectors of fixed $m^{1/2}$, occupying the sites of a hypercubic lattice in d spatial dimensions and the interactions J_{ij} are independent identically distributed random variables which couple nearest-neighbour pairs of sites (Edwards and Anderson 1975). As usual, the $\{J_{ij}\}$ are taken to be 'quenched' random variables, i.e. the free energy has to be calculated for fixed disorder and then an average taken over an ensemble of disorder configurations. In practice, the configuration average is usually carried out first, using the replica trick (Edwards and Anderson 1975). The use of this trick for symmetric replicas leads to a wrong result in the case of spin glasses with a finite number of components due to the lack of ergodicity. However, we believe this method provides an exact solution in the $m \to \infty$ limit. This idea is supported on one side, by numerical calculations (Morris *et al* 1986)

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which show that even for low values of d the number of minima of the Hamiltonian decreases rapidly as m increases and, on the other hand, the replica symmetric method gives an *exact* solution in the $m\to\infty$ limit of the long range m-vector model (Kosterlitz *et al* 1976).

The critical properties of this model have been studied in the high-temperature phase by Green *et al* (1982) who found that for $m=\infty$ there is an increase of the upper critical dimension, above which mean-field critical behaviour is expected, from six, for the finite m-vector spin-glass model (Harris *et al* 1976), to eight. The mechanism responsible for the shift in the value of the critical dimensionalities as $m\to\infty$ seems to be related to the fact that in this limit the quadrupole fields (to be defined later) play an important role as they couple different vector components. For m finite the quadrupole fields do not go 'soft' as they have a larger 'mass' than the other modes.

In the present work we consider the low-temperature phase. In order to solve this model we will make a calculation parallel to those of Bray and Moore (1979a, b), that is, we will make use of the replica method with a symmetric ansatz for the Ginzburg-Landau-Wilson (GLW) free-energy functional. In this case, we will work to cubic order $O(\mathbb{Q}^3)$ where $\mathbb{Q}^{\alpha\beta}_{\mu\nu}(x)$ is the order-parameter field, and calculate the spin-glass order parameter $\mathbb{Q}=\langle\mathbb{Q}^{\alpha\beta}_{\mu\nu}\rangle$ $(\alpha\neq\beta)$ to one-loop order. Pytte and Rudnick (1978) considered a truncated version of this free-energy functional which included terms to order $\mathbb{Q}^4$ and found instabilities which showed up as negative gaps in the correlation functions. The instabilities (negative gaps) of the $\mathbb{Q}^4$ model correspond to marginally stable solutions (gapless modes) of the $\mathbb{Q}^3$ model within mean-field theory.

2. The model

As we mentioned before, in this paper we will make use of the replica symmetric method in order to make analytical progress by averaging over the disorder at the outset. The price to pay for this mathematical simplification is that it will not be possible to take the $m\to\infty$ limit *ab initio*, but rather the calculation has to be performed for m finite and the limit taken at the end. In the replica method one calculates the free energy $-\lim_{n\to 0} kT(\langle Z^n\rangle_J-1)/n$ where $Z\{J_{ij}\}$ is the partition function of the system for a particular set of bonds $\{J_{ij}\}$ and $\langle\ \rangle_J$ denotes the average over all possible bond configurations which are distributed according to

$$P(J_{ij})=[\Delta(2\pi)^2]^{-1}\exp(-J_{ij}^2/2\Delta^2)$$

for i,j nearest neighbours and zero otherwise. For integer n one has

$$\langle Z^n\rangle_J=\int\prod_{\langle ij\rangle}\mathrm{d}J_{ij}P(J_{ij})\,\mathrm{Tr}\exp\Big((\beta/2)\sum_{ij\alpha\mu}J_{ij}S^{\alpha}_{i\mu}S^{\alpha}_{j\mu}\Big)$$

where $S_{i\mu}$ is the cartesian component μ of the spin S^{α}_i located at the site i and belonging to the replica α, and the trace is to be taken over all spin configurations. After averaging over all bond realisations we obtain

$$\langle Z^n\rangle_J=\mathrm{Tr}\exp\Big(\tfrac{1}{2}\sum_{ij}K_{ij}\sum_{\alpha\beta\mu\nu}S^{\alpha}_{i\mu}S^{\alpha}_{j\mu}S^{\beta}_{i\nu}S^{\beta}_{j\nu}\Big)$$

where $K_{ij}=\frac{1}{2}(\Delta/T)^2$ for i, j nearest neighbours and zero otherwise, and T is the temperature. In order to decouple the lattice sites we introduce a set $\{\ \ \}$ of auxiliary fields and obtain

$$\langle Z^n\rangle_J = \int \prod_{\substack{i\alpha\beta\\ \mu\nu}} \mathrm{d}\mathbb{Q}_{i\mu\nu}^{\alpha\beta} \exp\left\{ -\tfrac{1}{2} \sum_{\substack{ij\alpha\\ \beta\mu\nu}} (K_{ij})^{-1} \mathbb{Q}_{i\mu\nu}^{\alpha\beta} \mathbb{Q}_{j\mu\nu}^{\alpha\beta} + \ln\left[\mathrm{Tr}_\alpha \exp\left(\sum \mathbb{Q}_{i\mu\nu}^{\alpha\beta} S_{i\mu}^{\alpha} S_{i\nu}^{\beta}\right)\right]\right\}. \tag{1}$$

The order-parameter field $\mathbb{Q}_{i\mu\nu}^{\alpha\beta}$ has a part $\mathbb{Q}_{i\mu\nu}^{\alpha\beta}$ $(\alpha \neq \beta)$ which is off-diagonal in replica space and represents spin-glass order, and a diagonal part $\mathbb{Q}_{i\mu\nu}^{\alpha\alpha}$ which represents quadrupolar order. The trace (in spin space) of the latter field, i.e. $\sum_\mu \mathbb{Q}_{i\mu\mu}^{\alpha\alpha}$ represents a 'hard' or non-critical mode. It is convenient to separate the $\mathbb{Q}$ fields into their diagonal and off-diagonal parts, the latter being the spin-glass field of Harris *et al* (1976). The diagonal part may be written as

$$\mathbb{Q}_{i\mu\nu}^{\alpha\alpha} \to \mathbb{Q}_i^{\alpha\alpha} \delta_{\mu\nu} + T_{i\mu\nu}^{\alpha} \tag{2}$$

where $T_{\mu\nu}^{\alpha}$ is a traceless tensor which has been called the quadrupole field (Green *et al* 1982). With the decomposition (2), we can write

$$\sum_{\mu\nu} \mathbb{Q}_{i\mu\nu}^{\alpha\alpha} \mathbb{Q}_{j\mu\nu}^{\alpha\alpha} = m\mathbb{Q}_i^{\alpha\alpha}\mathbb{Q}_j^{\alpha\alpha} + \sum_{\mu\nu} T_{i\mu\nu}^{\alpha\alpha} T_{j\mu\nu}^{\alpha\alpha} \tag{3a}$$

$$\sum_{\mu\nu} \mathbb{Q}_{i\mu\nu}^{\alpha\alpha} S_{i\mu}^{\alpha} S_{i\nu}^{\alpha} = m\mathbb{Q}_i^{\alpha\alpha} + \sum_{\mu\nu} T_{i\mu\nu}^{\alpha\alpha} S_{i\mu}^{\alpha} S_{i\nu}^{\alpha}. \tag{3b}$$

Using (3) in (1), we find that the fields $\mathbb{Q}_i^{\alpha\alpha}$ decouple from the other fields and may be integrated out of the problem. Expanding the external exponent in (1) to third order in the spin-glass $\mathbb{Q}_{i\mu\nu}^{\alpha\beta}$ and the quadrupole $T_{i\mu\nu}^{\alpha\alpha}$ fields, carrying out the spin traces, re-exponentiating and taking the continuum limit yields a field theory

$$\langle Z^n\rangle_J = \int \mathrm{D}\mathbb{Q}(x) \exp\left(-\int F\{\mathbb{Q}\}\,\mathrm{d}x\right) \tag{4a}$$

where the measure $\mathrm{D}\mathbb{Q}$ includes both spin-glass and quadrupolar fields and F is the GLW free-energy density given by

$$F = \tfrac{1}{4}\sum [(\nabla \mathbb{Q}_{\mu\nu}^{\alpha\beta})^2 + r(\mathbb{Q}_{\mu\nu}^{\alpha\beta})^2] + \tfrac{1}{4}\sum\left[(\nabla T_{\mu\nu}^{\alpha}) + \left(r + \frac{\tau}{m+2}\right)(T_{\mu\nu}^{\alpha})^2\right] - \frac{w}{3!}\sum \mathbb{Q}_{\mu\nu}^{\alpha\beta}\mathbb{Q}_{\nu\rho}^{\beta\gamma}\mathbb{Q}_{\rho\mu}^{\gamma\alpha} - \frac{wm^2}{3!(m+2)(m+4)}\sum T_{\mu\nu}^{\alpha} T_{\nu\rho}^{\alpha} T_{\rho\mu}^{\alpha} - \frac{wm}{2!(m+2)}\sum T_{\mu\nu}^{\alpha}\mathbb{Q}_{\nu\rho}^{\alpha\beta}\mathbb{Q}_{\rho\mu}^{\alpha\beta} \tag{4b}$$

where τ and w are positive functions of the lattice spacing, the number of nearest neighbours and the temperature, and $r \propto (T - T_c)$. In this expression the sums are taken over all indices, and the restrictions $\mathbb{Q}_{\mu\nu}^{\alpha\alpha} = 0$, $\Sigma_\mu\ T_{\mu\mu}^{\alpha} = 0$ apply.

3. Mean-field theory

The mean-field solution can be found by using variational principles, i.e. by extremising the free-energy functional. We seek a replica symmetric solution of the form

$$\mathbb{Q}_{\mu\nu}^{\alpha\beta}(x) = \mathbb{Q}\,\delta_{\mu\nu}$$

$$T_{\mu\nu}^{\alpha}(x) = 0.$$

By introducing this solution into the free-energy functional and demanding $(1/n)\,\partial F/\partial \mathbb{Q}=0$ we find two stationary points given by $\mathbb{Q}=0$ and $\mathbb{Q}=r/[w(n-2)]$. In order to investigate the stability of these solutions against fluctuations, we need to expand the free-energy density about these stationary points. To this end we write

$$\mathbb{Q}^{\alpha\beta}_{\mu\nu}(x)=\mathbb{Q}\,\delta_{\mu\nu}+R^{\alpha\beta}_{\mu\nu} \tag{5a}$$

$$T^{\alpha}_{\mu\nu}(x)=S^{\alpha}_{\mu\nu} \tag{5b}$$

where $R^{\alpha\beta}_{\mu\nu}$ and $S^{\alpha}_{\mu\nu}$ are fluctuations around the mean-field solution satisfying the restrictions $R^{\alpha\alpha}_{\mu\nu}=0$, $\Sigma_\mu S^{\alpha}_{\mu\mu}=0$. If we substitute the Fourier transforms of equations (5) into (4), we get the effective Hamiltonian as a function of the fluctuations about the mean-field solution in the momentum space. Working in this space will be convenient since different Fourier components decouple to quadratic order. We can now calculate the correlation functions according to

$$\langle(\ )\rangle=\frac{\int \Pi\,\mathrm{d}R^{\alpha\beta}_{\mu\nu}\,\Pi\,\mathrm{d}S^{\alpha}_{\mu\nu}(\ )\exp[-F(R^{\alpha\beta}_{\mu\nu},S^{\alpha}_{\mu\nu})]}{\int \Pi\,\mathrm{d}R^{\alpha\beta}_{\mu\nu}\,\Pi\,\mathrm{d}S^{\alpha}_{\mu\nu}\exp[-F(R^{\alpha\beta}_{\mu\nu},S^{\alpha}_{\mu\nu})]} \tag{6}$$

where ( ) represents the correlation to be calculated and F is the free-energy functional. These correlations will have the form

$$\langle(\ )\rangle=\frac{1}{q^2+M^2}.$$

The stability condition requires the squared 'masses' M^2 to be positive since for $M^2<0$ the Gaussian integrals do not converge. In the case $\mathbb{Q}=0$, to quadratic order, the effective free energy takes the form

$$F=\tfrac{1}{4}\sum(q^2+r)(R^{\alpha\beta}_{\mu\nu})^2+\tfrac{1}{4}\sum[q^2+r+\tau/(m+2)](S^{\alpha}_{\mu\nu})^2.$$

Therefore there are two different correlation functions given by

$$\langle R^{\alpha\beta}_{\mu\nu}(q)R^{\alpha\beta}_{\mu\nu}(q)\rangle=\frac{1}{q^2+r}$$

$$\langle S^{\alpha}_{\mu\nu}(q)S^{\alpha}_{\mu\nu}(q)\rangle=\frac{1}{q^2+r+\tau/(m+2)}$$

which indicates that the solution $\mathbb{Q}=0$ is stable (or metastable) for $r>0$, i.e. in the high-temperature region $T>T_c$.

In a similar way we can see that the second solution given by $\mathbb{Q}=r/[w(2-n)]$ is unstable in the high-temperature region. In the low-temperature phase, namely for $r<0$, we have $\mathbb{Q}=|r|/[w(2-n)]$ and after using equations (5) we can write the resulting free energy as

$$F=\mathscr{F}_0-\mathscr{F}_1-\mathscr{F}_2$$

with

$$\mathscr{F}_0=\tfrac{1}{4}(q^2-|r|)\sum(R^{\alpha\beta}_{\mu\nu})^2+\tfrac{1}{4}[q^2+\tau/(m+2)-|r|]\sum(S^{\alpha}_{\mu\nu})^2 \tag{7a}$$

$$\mathscr{F}_1=\frac{|r|}{6(n-2)}\sum(R^{\alpha\beta}_{\mu\nu}R^{\beta\gamma}_{\nu\mu}+R^{\alpha\beta}_{\mu\nu}R^{\alpha\gamma}_{\mu\nu}+R^{\alpha\gamma}_{\mu\nu}R^{\beta\gamma}_{\mu\nu})$$
$$+\frac{|r|m}{2(m+2)(n-2)}\sum S^{\alpha}_{\mu\nu}R^{\alpha\beta}_{\mu\nu} \tag{7b}$$

$$\mathscr{F}_2 = \frac{w}{6}\sum R^{\alpha\beta}_{\mu\nu} R^{\beta\gamma}_{\nu\rho} R^{\gamma\alpha}_{\rho\mu} + \frac{wm^2}{6(m+2)(m+4)}\sum S^{\alpha}_{\mu\nu} S^{\alpha}_{\nu\rho} S^{\alpha}_{\rho\nu}$$

$$+\frac{wm}{2(m+2)}\sum S^{\alpha}_{\mu\nu} R^{\alpha\beta}_{\mu\rho} R^{\alpha\beta}_{\nu\rho}. \tag{7c}$$

By using $\mathscr{F}_0 - \mathscr{F}_1$ as the free-energy functional we will calculate exactly the correlation functions. Subsequently the perturbation $(-\mathscr{F}_2)$ will generate a systematic expansion in powers of w^2. The first step is to calculate the 'bare' propagators $\langle(\ )\rangle_0$ defined by using $\mathscr{F}_0$ as the free-energy functional in (6). In this case the $R^{\alpha\beta}_{\mu\nu}$ and the off-diagonal $S^{\alpha}_{\mu\nu}$ terms are completely decoupled and their correlation functions can be found by performing independent Gaussian integrals. In this way we obtain

$$\langle R^{\alpha\beta}_{\mu\nu}(q) R^{\alpha\beta}_{\mu\nu}(-q)\rangle_0 = \frac{1}{q^2-|r|} = g(q) \tag{8}$$

$$\langle R^{\alpha\beta}_{\mu\nu}(q) S^{\alpha}_{\mu\nu}(-q)\rangle_0 = \frac{1}{q^2-|r|+\tau/(m+2)} = f(q). \tag{9}$$

The diagonal S fields are not independent as they are coupled due to the traceless condition. For them we have

$$\begin{aligned}\langle S^{\alpha}_{\mu\mu}(q) S^{\alpha}_{\nu\nu}(-q)\rangle_0 &\\ &= \frac{2}{q^2+|r|+\tau/(m+2)}(\delta_{\mu\nu}-1/m)\\ &= 2f(q)(\delta_{\mu\nu}-1/m).\end{aligned} \tag{10}$$

These 'bare' propagators will be denoted as follows:

$$\langle R^{\alpha\beta}_{\mu\nu}(q) R^{\alpha\beta}_{\mu\nu}(-q)\rangle_0 = \underset{\mu\nu}{\overset{\alpha\beta}{\text{————————}}} \qquad (\alpha \neq 0)$$

$$\langle S^{\alpha}_{\mu\nu}(q) S^{\alpha}_{\mu\nu}(-q)\rangle_0 = \underset{\mu\nu}{\overset{\alpha\beta}{\sim\sim\sim\sim\sim}} \qquad (\mu \neq \nu)$$

$$\langle S^{\alpha}_{\mu\mu}(q) S^{\alpha}_{\nu\nu}(-q)\rangle_0 = \underset{\mu\mu-\nu\nu}{\overset{\alpha}{\sim\sim\sim\sim\sim}}.$$

If we now calculate the propagators by using $F = \mathscr{F}_0 - \mathscr{F}_1$ (7) as the 'free-energy functional' in (6), we will obtain the correlation functions calculated exactly. These 'dressed' propagators $\langle(\ )\rangle_{0-1}$ will be given by

$$\langle(\ )\rangle_{0-1} = \langle(\ )\exp(\mathscr{F}_1)\rangle_0 = \sum_{\nu=0}^{\infty}(1/\nu!)\langle(\ )(\mathscr{F}_1)^{\nu}\rangle_0$$

where $\langle(\ )\rangle_0$ are the 'bare' propagators given by equations (8)-(10) and the last sum runs over the 'connected' diagrams. There exist 23 different 'dressed' propogators divided into two decoupled groups that we will call diagonal and off-diagonal sectors. They will be considered separately.

3.1. Diagonal sector

The diagonal sector includes the following fourteen correlation functions:

$$
\begin{aligned}
G_1 &= \langle R^{\alpha\beta}_{\mu\mu}(q) R^{\alpha\beta}_{\mu\mu}(-q)\rangle & G_{11} &= \langle R^{\alpha\beta}_{\mu\mu}(q) R^{\alpha\beta}_{\nu\nu}(-q)\rangle \\
G_2 &= \langle R^{\alpha\beta}_{\mu\mu}(q) R^{\alpha\gamma}_{\mu\mu}(-q)\rangle & G_{21} &= \langle R^{\alpha\beta}_{\mu\mu}(q) R^{\alpha\gamma}_{\nu\nu}(-q)\rangle \\
G_3 &= \langle R^{\alpha\beta}_{\mu\mu}(q) R^{\gamma\delta}_{\mu\mu}(-q)\rangle & G_{31} &= \langle R^{\alpha\beta}_{\mu\mu}(q) R^{\gamma\delta}_{\nu\nu}(-q)\rangle \\
G_4 &= \langle S^{\alpha}_{\mu\mu}(q) S^{\alpha}_{\mu\mu}(-q)\rangle & G_{41} &= \langle S^{\alpha}_{\mu\mu}(q) S^{\alpha}_{\nu\nu}(-q)\rangle \\
G_5 &= \langle S^{\alpha}_{\mu\mu}(q) S^{\beta}_{\mu\mu}(-q)\rangle & G_{51} &= \langle S^{\alpha}_{\mu\mu}(q) S^{\beta}_{\nu\nu}(-q)\rangle \\
G_6 &= \langle R^{\alpha\beta}_{\mu\mu}(q) S^{\alpha}_{\mu\mu}(-q)\rangle & G_{61} &= \langle R^{\alpha\beta}_{\mu\mu}(q) S^{\alpha}_{\nu\nu}(-q)\rangle \\
G_7 &= \langle R^{\alpha\beta}_{\mu\mu}(q) S^{\gamma}_{\mu\mu}(-q)\rangle & G_{71} &= \langle R^{\alpha\beta}_{\mu\mu}(q) S^{\gamma}_{\nu\nu}(-q)\rangle .
\end{aligned}
$$

In all of them $\alpha \neq \beta \neq \gamma \neq \delta$ and $\mu \neq \nu$. Figure 1 shows the Dyson-type equations corresponding to two typical propagators for n finite. In these graphs double lines represent the 'dressed' propagators, the open (○) and full (•) circles carry the factors $|r|/(2-n)$ and $|r|m/(2-n)(m+2)$ respectively, and the coefficient corresponding to each graph indicates the existence of other graphs with the same value as the graphs shown but different labelling. The remaining Dyson-type equations can be found elsewhere (Viana 1985). After taking the $n \to 0$ limit, the diagonal propagators satisfy the following equations (where the dependence on q has not been written explicitly):

$$
\begin{aligned}
G_1 &= g\{1 + \circ[-4G_2] + \bullet[2G_6]\} \qquad G_2 = g\{\circ[G_1 - 2G_2 - 3G_3] + \bullet[G_6 + G_7]\} \\
G_3 &= g\{\circ[4G_2 - 8G_3] + \bullet[2G_7]\} \qquad G_4 = 2f(1-1/m)\{1 - \bullet[G_6 - G_{61}]\} \\
G_5 &= 2f(1-1/m)\bullet\{[G_6 - G_{61}] - 2[G_7 - G_{71}]\} \\
G_6 &= g\{\circ[-2(G_6 + G_7)] + \bullet[G_4 + G_5]\} = 2f(1-1/m)\bullet\{[G_1 - G_{11}] - 2[G_2 - G_{21}]\} \\
G_7 &= g\{\circ[2G_6 - 6G_7] + \bullet[2G_5]\} = 2f(1-1/m)\bullet\{-3[G_3 - G_{31}] + 2[G_2 - G_{21}]\} \\
G_{11} &= g\{\circ[-4G_{21}] + \bullet[2G_{61}]\} \qquad G_{21} = g\{\circ[G_{11} - 2G_{21} - 3G_{31}] + \bullet[G_{61} + G_{71}]\} \\
G_{31} &= g\{\circ[4G_{21} - 8G_{31}] + \bullet[2G_{71}]\} \qquad G_{41} = -(2f/m)\{1 - \bullet[G_6 - G_{61}]\}
\end{aligned}
$$

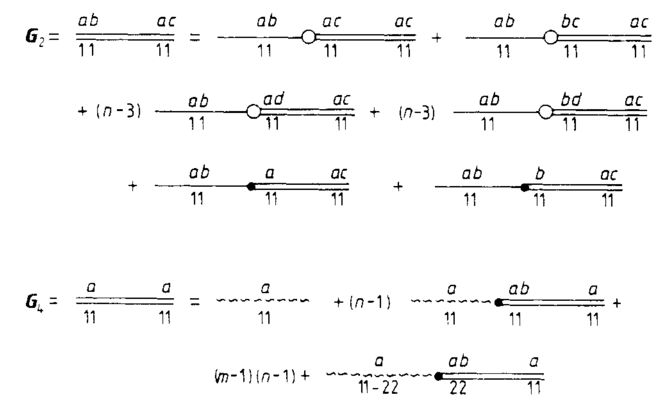

Figure 1. Diagrammatic expansion for the propagators G_2 and G_4. Single and double lines represent 'bare' and 'dressed' propagators respectively. The open (○) and full (•) circles carry the factors $|r|/(2-n)$ and $|r|m/(2-n)(m+2)$ respectively.

$$G_{51} = -(2f/m)\bullet\{[G_6 - G_{61}] - 2[G_7 - G_{71}]\}$$

$$G_{61} = g\{\bigcirc[-2(G_{61} + G_{71})] + \bullet[G_{41} + G_{51}]\} = -(2f/m)\bullet\{[G_1 - G_{11}] - 2[G_2 - G_{21}]\}$$

$$G_{71} = g\{\bigcirc[2G_{61} - 6G_{71}] + \bullet[2G_{51}]\} = (2f/m)\bullet\{3[G_3 - G_{31}] - 2[G_2 - G_{21}]\}$$

with g and f given by equations (8) and (9). If we make the change of variables

$$\mathbb{G}_i = G_i + (m-1)G_{i1} \tag{11}$$

$$X_i = G_i - G_{i1} \tag{12}$$

for $i = 1, \ldots, 7$, then these equations decouple into two sets $\{\mathbb{G}_i\}$ and $\{X_i\}$. By solving the equations for the first set we find

$$\mathbb{G}_1 = \frac{q^4 + 3q^2|r| + 3|r|^2}{q^2(q^2+|r|)^2} \tag{13}$$

$$\mathbb{G}_2 = \frac{|r|}{2q^2}\frac{(q^2+3|r|)}{(q^2+|r|)^2} \tag{14}$$

$$\mathbb{G}_3 = \frac{|r|^2}{q^2(q^2+|r|)^2} \tag{15}$$

$$\mathbb{G}_i = 0 \qquad i = 4, \ldots, 7. \tag{16}$$

There are some simple combinations of these propagators which have simple poles in the q plane and therefore correspond to 'pure' modes. These combinations are given by

$$\mathbb{G}_B = \mathbb{G}_1 - 4\mathbb{G}_2 + 3\mathbb{G}_3 = 1/(q^2+|r|)$$

$$\mathbb{G}_R = \mathbb{G}_1 - 2\mathbb{G}_2 + \mathbb{G}_3 = 1/q^2.$$

We have denoted them $\mathbb{G}_B$ and $\mathbb{G}_R$ because they are the infinite-vector analogues to the 'breathing mode' and the 'replicon mode' for the finite m case (see Bray and Moore (1979a, b) for a discussion). The remaining seven propagators $\{X_i\}$ are found to be given in order $1/m$ by

$$X_1 = \frac{1}{q^6}(q^4 + q^2|r| + |r|^2) + \frac{1}{m}\left(\frac{1}{q^{10}}[q^4(-4|r|^2) + q^2(-4|r|^3 - |r|^2\tau) + (8|r|^4 - 2|r|^3\tau)]\right) \tag{17}$$

$$X_2 = \frac{|r|}{2q^6}(q^2 + 2|r|) + \frac{1}{m}\left(\frac{1}{q^{10}}[q^4(-2|r|^2) + q^2(-6|r|^3 - |r|^2\tau/2) + (8|r|^4 - 2|r|^3\tau)]\right) \tag{18}$$

$$X_3 = \frac{|r|^2}{q^6} + \frac{1}{m}\left(\frac{1}{q^{10}}[q^2(-8|r|^3) + (8|r|^4 - 2|r|^3\tau)]\right) \tag{19}$$

$$\begin{aligned} X_4 = {} & \frac{1}{q^6}[2q^4 + q^2(2|r|) + |r|^2] \\ & + \frac{1}{m}\left(\frac{1}{q^{10}}[q^6(-2\tau) + q^4(4|r|^2 - 4|r|\tau) \right. \\ & \left. + q^2(8|r|^3 - 4|r|^2\tau) + (8|r|^4 - 2|r|^3\tau)]\right) \end{aligned} \tag{20}$$

$$X_5=\frac{|r|^2}{q^6}+\frac{1}{m}\left(\frac{1}{q^{10}}[q^4(-4|r|^2)+q^2(-2|r|^2\tau)+(8|r|^4-2|r|^3\tau)]\right) \tag{21}$$

$$X_6=\frac{|r|}{q^6}(q^2+|r|)$$
$$+\frac{1}{m}\left(\frac{1}{q^{10}}[q^6(-2|r|)+q^4(-2|r|^2-|r|\tau)+q^2(-2|r|^2\tau)+(8|r|^4-2|r|^3\tau)]\right) \tag{22}$$

$$X_7=\frac{|r|^2}{q^6}+\frac{1}{m}\left(\frac{1}{q^{10}}[q^4(-2|r|^2)+q^2(-4|r|^3-|r|^2\tau)+(8|r|^4-2|r|^3\tau)]\right). \tag{23}$$

There is one combination of propagators X corresponding to a 'pure' mode; this is given by

$$X_R=X_1-2X_2+X_3=1/q^2.$$

The values of G_i, G_{i1} $(i=1,\ldots,7)$ can now be obtained by inverting equations (11) and (12), i.e.

$$G_i=\frac{1}{m}\mathbb{G}_i+\left(1-\frac{1}{m}\right)X_i$$

$$G_{i1}=\frac{1}{m}[\mathbb{G}_i-X_i]$$

with $\mathbb{G}_i$ and X_i given by (13)-(16) and (17)-(23) respectively. From the result obtained we can see that in the $m\to\infty$ limit all these propagators are either massless or zero.

3.2. *Off-diagonal sector*

The off-diagonal sector includes the propagators

$$G_{12}=\langle R^{\alpha\beta}_{\mu\nu}(q)R^{\alpha\beta}_{\mu\nu}(-q)\rangle \qquad G_{22}=\langle R^{\alpha\beta}_{\mu\nu}(q)R^{\alpha\gamma}_{\mu\nu}(-q)\rangle$$
$$G_{32}=\langle R^{\alpha\beta}_{\mu\nu}(q)R^{\gamma\delta}_{\mu\nu}(-q)\rangle \qquad G_{42}=\langle S^{\alpha}_{\nu\mu}(q)S^{\alpha}_{\mu\nu}(-q)\rangle$$
$$G_{52}=\langle S^{\alpha}_{\mu\nu}(q)S^{\beta}_{\mu\nu}(-q)\rangle \qquad G_{62}=\langle R^{\alpha\beta}_{\mu\nu}(q)S^{\alpha}_{\mu\nu}(-q)\rangle$$
$$G_{72}=\langle R^{\alpha\beta}_{\mu\nu}(q)S^{\gamma}_{\mu\nu}(-q)\rangle \qquad G_{13}=\langle R^{\alpha\beta}_{\mu\nu}(q)R^{\alpha\beta}_{\nu\mu}(-q)\rangle$$
$$G_{23}=\langle R^{\alpha\beta}_{\mu\nu}(q)R^{\alpha\gamma}_{\nu\mu}(-q)\rangle$$

which satisfy Dys n-type equations similar to those of the diagonal sector (see Viana (1985)). Following the same procedure as before we find for them in the $n\to 0$ limit

$$G_{12}=g\{1+\bigcirc[-4G_{22}]+\bullet[2G_{62}]\}$$
$$G_{22}=g\{\bigcirc[-3(G_{22}+G_{32})+(G_{12}+G_{23})]+\bullet[G_{62}+G_{72}]\}$$
$$G_{32}=g\{\bigcirc[-8G_{32}+2G_{23}+2G_{22}]+\bullet[2G_{72}]\}$$
$$G_{42}=f\{1-\bullet[2G_{62}]\} \qquad G_{52}=f\bullet[-4G_{72}+2G_{62}]$$
$$G_{62}=f\bullet[-2(G_{22}+G_{23})+G_{12}+G_{13}]=g\{\bigcirc[-2(G_{62}+G_{72})]+\bullet[G_{42}+G_{52}]\}$$
$$G_{72}=f\bullet[-6G_{32}+2G_{23}+2G_{22}]=g\{\bigcirc[-6G_{72}+2G_{62}]+\bullet[2G_{52}]\}$$
$$G_{13}=g\{\bigcirc[-4G_{23}]+\bullet[2G_{62}]\}$$
$$G_{23}=g\{\bigcirc[-3(G_{23}+G_{32})+G_{13}+G_{22}]+\bullet[G_{62}+G_{72}]\}.$$

If we now make the substitutions

$$F_1 = G_{12} + G_{13} \qquad F_2 = G_{22} + G_{23}$$
$$F_i = 2G_{i2} \qquad i = 3, \ldots, 7$$

we can see that these new variables $\{F_i\}$ satisfy the same equations as $\{X_i\}$. Therefore, $F_i = X_i$ for $i = 1, \ldots, 7$ with X_i given by equations (17)-(23). It is also convenient to define

$$H_1 = G_{12} - G_{13} \qquad H_2 = G_{22} - G_{23}.$$

These combinations correspond to 'massless' modes as they satisfy

$$H_1 = \frac{q^2 + |r|}{q^4} \tag{24}$$

$$H_2 = \frac{|r|}{2q^4}. \tag{25}$$

Now it is easy to obtain the propagators included in the off-diagonal sector by using the relations

$$G_{i2} = \tfrac{1}{2}[X_i + H_i] \qquad i = 1, 2$$
$$G_{i3} = \tfrac{1}{2}[X_i - H_i] \qquad i = 1, 2$$
$$G_{i2} = \tfrac{1}{2}X_i \qquad i = 3, \ldots, 7$$

with X_i and H_i given by equations (17)-(23), and (24) and (25) respectively. It is important to notice that all correlation functions were found to be massless.

4. Perturbation theory

In this section we will make a one-loop expansion to see how the order parameter $\langle Q^{\alpha\beta}_{\mu\nu}\rangle$ is affected by the presence of the perturbation $(\mathcal{F}_2)$ of equation (7*c*). From (9) we have

$$\langle Q^{\alpha\beta}_{\mu\nu}\rangle = \frac{|r|}{2w}\,\delta_{\mu\nu} + \langle R^{\alpha\beta}_{\mu\nu}\rangle.$$

The value of $\langle R^{\alpha\beta}_{\mu\nu}\rangle$ due to the cubic term is given exactly by

$$\langle R^{\alpha\beta}_{\mu\nu}\rangle_{0\text{-}1\text{-}2} = \sum_{\nu=0}^{\infty} \frac{1}{\nu!}\langle R^{\alpha\beta}_{\mu\nu}(\mathcal{F}_2)^{\nu}\rangle_{0\text{-}1}$$

where $\langle(\ )\rangle_{0\text{-}1}$ are the 'dressed' propagators just calculated and $\mathcal{F}_2$ is given by (7*c*). To lowest non-trivial order we get

$$\begin{aligned}\langle R^{\alpha\beta}_{\mu\mu}(q)\rangle_{0\text{-}1\text{-}2} = {} & \langle R^{\alpha\beta}_{\mu\mu}(q)\rangle_{0\text{-}1} + \frac{w}{6}\left\langle R^{\alpha\beta}_{\mu\mu}(q)\sum R^{\varepsilon\eta}_{\lambda\nu}(q_1)R^{\eta\gamma}_{\nu\rho}(q_2)R^{\gamma\varepsilon}_{\rho\lambda}(-q_1-q_2)\right\rangle_{0\text{-}1}\\ & + \frac{wm^2}{6(m+2)(m+4)}\left\langle R^{\alpha\beta}_{\mu\mu}(q)\sum S^{\gamma}_{\lambda\nu}(q_1)S^{\gamma}_{\nu\rho}(q_2)S^{\gamma}_{\rho\lambda}(-q_1-q_2)\right\rangle_{0\text{-}1}\\ & + \frac{wm}{2(m+2)}\left\langle R^{\gamma\beta}_{\mu\mu}(q)\sum S^{\gamma}_{\lambda\nu}(q_1)R^{\gamma\varepsilon}_{\lambda\rho}(q_2)R^{\gamma\varepsilon}_{\nu\rho}(-q_1-q_2)\right\rangle_{0\text{-}1}\end{aligned}$$

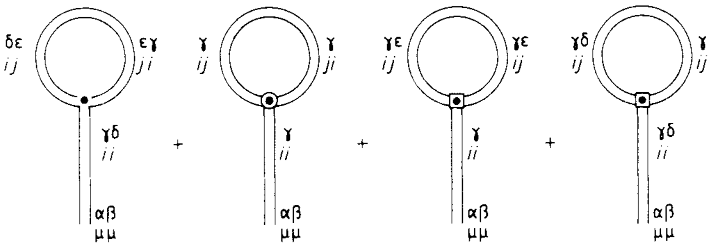

Figure 2. Graphs for $\langle R_{\mu\mu}\rangle$, the deviation of $\langle Q_{\mu\mu}\rangle$ from its mean-field value, to the lowest order in perturbation theory. The terms •, ⊙ and ⊡ represent a cubic interaction and carry the factors $w/2$, $wm^2/2(m+2)(m+4)$ and $wm/2(m+2)$ respectively. The expression has to be summed over dummy replica and spin indices.

where α, β, μ are fixed and the remaining indices are to be summed over. This expression can be worked out with the aid of diagrams, as shown in figure 2 where double lines represent 'dressed' propagators, and has to be summed over dummy replica and spin indices. In these diagrammatic expressions the signs •, ⊙ and ⊡ carry the factors $w/2$, $wm^2/2(m+2)(m+4)$ and $wm/2(m+2)$ respectively. The external legs carry $q=0$ and closed loops represent the summation over all q. After taking the $n\to 0$ limit, we obtain

$$\langle R^{\alpha\beta}_{\mu\mu}(0)\rangle = \mathbf{G}_B|_{q=0}\left(-2w\sum_q[G_2(q)+(m-1)G_{22}(q)] + \frac{2w}{1+(2/m)}\sum_q[G_6(q)+(m-1)G_{62}(q)]\right).$$

After substituting the values for the correlation functions we obtain

$$\langle R^{\alpha\beta}_{\mu\mu}(0)\rangle = \frac{w}{|r|}\sum_q\left(\frac{1}{q^8}[q^4(-3|r|/2)+q^2(-|r|^2-|r|\tau/2)+(3|r|^3-3|r|^2\tau/4)]\right)+\mathrm{O}(1/m).$$

We can see that this expression diverges for $d\leqslant 8$ dimensions.

5. Discussion

We have found that perturbation theory for the order parameter reveals, at one-loop order, very unusual properties. On one hand, the perturbation expansion fails due to the appearance of infrared singularities for all dimensionalities $d\leqslant 8$ while $d_c=8$ is believed to be the upper critical dimensionality of the theory. On the other hand, it was expected that the temperature dependence of Q near T_c would be changed from its mean-field form by terms which are logarithmic in $(T-T_c)$ for $d=8$. Standard methods (Wilson 1972) would then allow, via the exponentiation of the logarithm, the determination of the critical exponent β (defined by $Q\propto(T_c-T)^\beta$) to order ε. However, the expected logarithmic terms were found to be absent, which seems to imply a breakdown of scaling. The interpretation of these results is unclear at present.

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